On the Expansion of any Functions of Multinomials. By Thomas Knight, *Esq. Communicated by* Humphry Davy, *Esq. LL.D. Sec. R. S.*

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1. THE expansion of multinomial functions has, of late, been so ably and fully treated by M. ARBOGAST, in his learned work ' *Du Calcul des Dérivations,*' that it may appear, perhaps, scarcely necessary to add any thing to what has been written, on the subject, by that excellent geometer.

Nevertheless, as he is the only one that has hitherto cultivated this part of analysis with any *great* success ; and as it is agreeable, I believe, to most persons, to be presented with various solutions to mathematical problems, I hope it will not be thought superfluous if I show how the same things may be accomplished in a very different manner.

By the procedure here made use of, we shall also be enabled to arrive at many new and remarkable theorems (both for *direct* and *inverse derivation*), which could not, I imagine, be very easily found by M. ARBOGAST's methods.

For a function of *one* simple multinomial, I give (amongst others) the same rules of *direct derivation,* as that author ; but when there are *many,* and in the more difficult cases of double and triple multinomials, &c. or functions of any number of these, I offer new and expeditious methods ; which are demonstrated with the less trouble, from the analogy which

reigns throughout, in this manner of treating the subject; and the regularity with which we proceed from the easy to the more complex cases. By means of this analogy also, the reader may without difficulty keep all the rules in his memory.

2. I shall begin with *the expansion of any function of a simple multinomial.*

*First method.** If $f(c + z)$ represent any function of $c + z$, and the fluxions be taken, separately, with respect to c and z, the fluxional coefficient is the same in both cases: or

$$\left(\overline{\frac{\dot{f(c+z)}}{\dot{c}}}\right) = \left(\overline{\frac{\dot{f(c+z)}}{\dot{z}}}\right); \text{ whence it follows, that}$$

$$\int\left(\overline{\frac{\dot{f(c+z)}}{\dot{c}}}\right)\dot{z} = \int\left(\overline{\frac{\dot{f(c+z)}}{\dot{z}}}\right)\dot{z} = f(c+z). \quad \text{This being pre-}$$

mised, let

$$f(c + \dot{c}x + \overset{\shortmid\shortmid}{c}x^2 + \overset{\shortmid\shortmid\shortmid}{c}x^3 +) = B + \underset{1}{B}x + \underset{2}{B}x^2 + \underset{3}{B}x^3 + \dots$$

$$+ \underset{n}{B}x^n +, \&c. \dots (1);$$

let $\overset{\shortmid}{\underset{1}{B}}, \overset{\shortmid}{\underset{2}{B}}, \overset{\shortmid}{B}$, &c. represent the fluxional coefficients of B, $\underset{1}{B}$, $\underset{2}{B}$, &c. with respect to c, and we shall have

$$\left(\overline{\frac{\dot{f(c + \overset{\shortmid}{c}x + \overset{\shortmid\shortmid}{c}x^2 + \overset{\shortmid\shortmid\shortmid}{c}x^3 +)}}{\dot{c}}}\right) = \overset{\shortmid}{B} + \overset{\shortmid}{\underset{1}{B}}x + \overset{\shortmid}{\underset{2}{B}}x^2 + \overset{\shortmid}{\underset{3}{B}}x^3 + \dots + \overset{\shortmid}{\underset{n}{B}}x^n +.$$

If we multiply this by $\dot{c}\dot{x} + 2\overset{\shortmid\shortmid}{c}x\dot{x} + 3\overset{\shortmid\shortmid\shortmid}{c}x^2\dot{x} + \dots + n\overset{\shortmid\shortmid..n}{c}$ $x^{n-1}\dot{x} +$ and take the fluent, we shall get, by what was just now shewn, another expansion of $f(c + \dot{c}x + \overset{\shortmid\shortmid}{c}x^2 + \overset{\shortmid\shortmid\shortmid}{c}x^3 +)$;

* See LA CRIOX, ' *Traité élément. de Calc. Différent.*' p. 25, note; where a similar proceeding is used for binomial functions.

and by comparing the coefficients, of the different powers of x, with those in equation (1), there will be found,

$$B_1 = c' \, B'$$

$$B_2 = \frac{2\,c''\, B' + c'\, B'_1}{2}$$

$$B_3 = \frac{3\,c'''\, B' + 2\,c''\, B'_1 + c'\, B'_2}{3}$$

. . . .

$$B_n = \frac{n\;\overset{''...n}{c}\;B' + (n-1)\;\overset{''..(n-1)}{c}\;B'_1 + (n-2)\;\overset{''..(n-2)}{c}\;B'_2 + ... + 2\,c''\, B'_{n-2} + c'\, B'_{n-1}}{n}$$

..... (2).

But $B = f(c), B' = f'(c)$, whence all the rest are known. I represent by strokes over the f the fluxional coefficients of $f(c)$; the number of strokes marking the order.

Though this is a complete solution of the problem, it affords by no means an easy way of calculating the coefficients; on which account I shall not trouble the reader with examples. It will be shewn presently, that the method of derivation in M. ARBOGAST's first section is easily obtained from this.

3. *Second Method.* I here, as in the former case, consider the quantity $c + c'\, x + c''\, x^2 + c'''\, x^3 +$ to be a binomial, and take the fluxional coefficient of the function with respect to c; but multiply by the partial fluxion $c' \dot{x}$, instead of $c' \dot{x} + 2 c'' x \dot{x}$

+, &c.; we find, by this way of proceeding, $\int_{n-1}^{B} c'$ for the sum of all those terms in B_n that are multiplied by the powers

of $\overset{\prime}{c}$. In like manner, $\int_{n-2} \overset{\prime}{B} \overset{\prime\prime}{c}$ will be the sum of all those

terms, in the same coefficient, that contain $\overset{\prime\prime}{c}$ and its powers;

and, in general, $\int_{n-m} \overset{\prime}{B} \overset{\prime\prime...m}{c}$ the sum of those that have for factors

the powers of $\overset{\prime\prime...m}{c}$.

Hence is derived an easy method of finding any coefficient, when we know those that precede it: for if these partial values be united, there arises

$$\underset{n}{B} = \int_{n-1} \overset{\prime}{B} \overset{\prime}{c} + \int_{n-2} \overset{\prime}{B} \overset{\prime\prime}{c} + \int_{n-3} \overset{\prime}{B} \overset{\prime\prime\prime}{c} +, \&c. \dots\dots (3), \text{ provided that}$$

we neglect

     in $\underset{n-2}{B}$ all those terms which contain $\overset{\prime}{c}$,

     in $\underset{n-3}{B}$ all those which contain   - $\overset{\prime}{c}$ or $\overset{\prime\prime}{c}$,

     in $\underset{n-4}{B}$ all those which have   - $\overset{\prime}{c}$ or $\overset{\prime\prime}{c}$ or $\overset{\prime\prime\prime}{c}$,

and so on; whence it happens, that many of the B's will be neglected entirely, and the chief part of the operation will

always be in the first term $\int_{n-1} \overset{\prime}{B} \overset{\prime}{c}$. From equation (3), we

find the first part of the expansion of $f(c + \overset{\prime}{c} x + \overset{\prime\prime}{c} x^2 +)$
to be

$$
\begin{aligned}
f(\overset{*}{c}) + f'(c)\overset{\prime}{c}x &+ f'(c)\overset{\prime\prime}{c}\bigg|x^2 + f'(c)\overset{*\prime\prime\prime}{c}\bigg|x^3 + f'(c)\overset{*\prime\prime\prime\prime}{c}\bigg|x^4 + f'(c)\overset{\prime\prime\prime\prime\prime}{c}\bigg|x^5 \\
&+ f''(c)\frac{\overset{\prime}{c}{}^2}{2}\bigg| + f''(c)\overset{\prime}{c}\overset{\prime\prime}{c}\bigg| + f''(c)\left\{\overset{\prime}{c}\overset{\prime\prime\prime}{c} + \frac{\overset{\prime\prime}{c}{}^2}{2}\right\} + f''(c)\left\{\overset{\prime}{c}\overset{\prime\prime\prime\prime}{c} + \overset{\prime\prime}{c}\overset{\prime\prime\prime}{c}\right\} \\
&\qquad + f'''(c)\frac{\overset{\prime}{c}{}^3}{2.3}\bigg| + f'''(c)\frac{\overset{\prime}{c}{}^2}{2}\overset{\prime\prime}{c} + f'''(c)\left\{\frac{\overset{\prime}{c}{}^2}{2}\overset{\prime\prime\prime}{c} + \overset{\prime}{c}\frac{\overset{\prime\prime}{c}{}^2}{2}\right\} \\
&\qquad\qquad + f''''(c)\frac{\overset{\prime}{c}{}^4}{2.3.4} + f''''(c)\frac{\overset{\prime}{c}{}^3}{2.3}\overset{\prime\prime}{c} \\
&\qquad\qquad\qquad + f'''''(c)\frac{\overset{\prime}{c}{}^5}{2.3.4.5}
\end{aligned}
$$

But, that we may enter rather more into particulars, let it be required from the terms already given, to find $\overset{B}{_6}$ the coefficient of x^6.

To make the operation plain, I have put a star over every term we are to use, excepting the coefficient of x^5, which is wholly employed.

$$\int_{6-1}^{\overset{\prime B}{}} \overset{\prime}{c} = \int_{5}^{\overset{\prime B}{}} \overset{\prime}{c} = f'(c)\,\overset{\prime}{c}\,\overset{\prime\prime\prime\prime}{c} + f''(c)\left|\frac{\overset{\prime2}{c}}{2}\overset{\prime\prime\prime\prime}{c} + f'''(c)\right|\frac{\overset{\prime3}{c}}{2.3}\overset{\prime\prime\prime}{c} + f''''(c)\frac{\overset{\prime4}{c}}{2.3.4}\overset{\prime\prime}{c} + f'''''(c)\frac{\overset{\prime6}{c}}{2.3.4.5.6};$$
$$+\,\overset{\prime}{c}\,\overset{\prime\prime}{c}\,\overset{\prime\prime\prime}{c} \qquad + \frac{\overset{\prime2}{c}\,\overset{\prime\prime}{c}^2}{2.2}$$

$$\int_{4}^{\overset{\prime B}{}} \overset{\prime\prime}{c} = f''(c)\,\overset{\prime\prime}{c}\,\overset{\prime\prime\prime\prime}{c} + f'''(c)\,\frac{\overset{\prime\prime}{c}^3}{2.3}; \quad \int_{3}^{\overset{\prime B}{}} \overset{\prime\prime\prime}{c} = f'''(c)\,\frac{\overset{\prime\prime}{c}^2}{2}; \int \overset{\prime B}{} \overset{\prime\prime\prime\prime}{c} = f''''(c)\,\overset{\prime\prime\prime...6}{c};$$

and by adding these together, we get

$$\overset{B}{_6} = f'(c)\,\overset{\prime\prime...6}{c} + f''(c)\left|\overset{\prime}{c}\,\overset{\prime\prime\prime\prime}{c} + f'''(c)\right|\frac{\overset{\prime2}{c}}{2}\overset{\prime\prime\prime\prime}{c} + f''''(c)\left|\frac{\overset{\prime3}{c}}{2.3}\overset{\prime\prime\prime}{c} + f'''''(c)\right|\frac{\overset{\prime4}{c}}{2.3.4}\overset{\prime\prime}{c} + f''''''(c)\frac{\overset{\prime6}{c}}{2.3.4.5.6}.$$
$$+\,\overset{\prime\prime}{c}\,\overset{\prime\prime\prime\prime}{c} \qquad +\,\overset{\prime}{c}\,\overset{\prime\prime}{c}\,\overset{\prime\prime\prime}{c} \qquad +\frac{\overset{\prime2}{c}\,\overset{\prime\prime}{c}^2}{2.2}$$
$$+\frac{\overset{\prime\prime\prime}{c}^2}{2} \qquad +\frac{\overset{\prime\prime}{c}^3}{2.3}$$

This process is sufficiently easy; but, in order to find any coefficient as $\overset{B}{_n}$, it is by no means necessary for us to know all those that precede it; it may be immediately obtained from $\overset{B}{_{n-1}}$ by a variety of ways: but we must first learn how to express $\overset{\prime B}{_{n-m}}$ by the fluxional coefficients of $\overset{B}{_{n-1}}$; after which we shall only have to substitute in equations (2) and (3).

Now it appears, from what has been shewn in this article, that

$$\overset{\prime B}{_{n-1}} = \left(\frac{\overset{\centerdot B}{_n}}{\overset{\prime}{c}}\right); \quad \overset{\prime B}{_{n-2}} = \left(\frac{\overset{\centerdot B}{_n}}{\overset{\prime\prime}{c}}\right) = \left(\frac{\overset{\centerdot B}{_{n-1}}}{\overset{\prime}{c}}\right); \quad \overset{\prime B}{_{n-3}} = \left(\frac{\overset{\centerdot B}{_n}}{\overset{\prime\prime\prime}{c}}\right) = \left(\frac{\overset{\centerdot B}{_{n-1}}}{\overset{\prime\prime}{c}}\right)$$

$$= \left(\frac{\overset{\centerdot\centerdot B}{_{n-2}}}{\overset{\prime}{c}}\right) = \left(\frac{\overset{\centerdot\centerdot B}{_{n-1}}}{\overset{\prime}{c}^2}\right); \quad \text{(where by strokes put } \textit{under} \text{ a quan-}$$

tity, I represent the reverse of the operations denoted by the strokes *over it*)* and, in general,

$$\underset{n-m}{\overset{'}{B}} = \left(\dfrac{\underset{n}{\overset{\cdot}{B}}}{\underset{c}{\overset{''..m}{}} \cdot} \right) = \left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c}{\overset{''..(m-1)}{}} \cdot} \right) =, \&c. \;\; ....(4); \;\; \underset{n-m}{\overset{'}{B}} = \left(\dfrac{\overset{\big(\overset{m-1}{\overset{\cdot}{B}}\big)}{\underset{n-1}{}}}{\underset{c \cdot}{,\,m-1}} \right)_{''......(m-2)} ......(5).$$

It is evident that we might find many other relations between the B's and their fluxional coefficients; but those I have given seem the most useful.

4. By means of these equations, we may find $\underset{n}{B}$ from $\underset{n-1}{B}$ in several ways.

First Method. If we substitute, in equation (2), for $\underset{n-2}{\overset{'}{B}}$,

$\left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c\cdot}{'}} \right)$; for $\underset{n-3}{\overset{'}{B}}$, $\left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c\cdot}{''}} \right)$; for $\underset{n-4}{\overset{'}{B}}$, $\left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c\cdot}{'''}} \right)$; .... for $\underset{n-n}{\overset{'}{B}}$,

$\left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c}{\overset{''..n(-1)}{}} \cdot} \right)$; which values are got from equation (4), we find

$$n \underset{n}{B} = n \overset{''...n}{c} \left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c}{\overset{''..(n-1)}{}} \cdot} \right) + (n-1) \overset{''...(n-1)}{c} \left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c}{\overset{''...(n-2)}{}} \cdot} \right) + ...$$

$$+ 2 \overset{''}{c} \left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{\underset{c\cdot}{'}} \right) + \overset{'}{c} \left(\dfrac{\underset{n-1}{\overset{\cdot}{B}}}{c} \right) .... (6).$$

This expression agrees with M. ARBOGAST's first method, and affords the following rule.

To find $\underset{n}{B}$ *from* $\underset{n-1}{B}$, *take the fluxion of the latter, with respect to* $c, \overset{'}{c}, \overset{''}{c}$, *&c. and change generally* $\overset{''...m}{c} \cdot$ *into* $\overset{''...(m+1)}{c} \times \dfrac{m+1}{n}$.

* Any number of strokes under a quantity will represent the depression of the fluxional coefficients of + (*c*) therein contained so many orders.

This rule is, however, more simple in the enunciation than the practice; on which account I proceed to a

5. *Second Method.* We might obtain one from equations (2) and (5), but, as it would be somewhat worse than the last, I omit it; and substitute in equation (3), the values of $\overset{\prime}{\underset{n-2}{B}}$, $\overset{\prime}{\underset{n-3}{B}}$, &c. given by (5). We find thus

$$\underset{n}{B} = \int\left(\frac{\overset{\cdot}{\underset{n-1}{B}}}{\overset{\cdot}{c}}\right)\overset{\prime}{c} + \int\left(\frac{\overset{\cdot}{\underset{n-1}{B}}}{\overset{\cdot}{c}}\right)\overset{\prime\prime}{c} + \int\left(\frac{\overset{\cdot\cdot}{\underset{n-1}{B}}}{\overset{\cdot}{c}^2}\right)\overset{\prime\prime\prime}{c} + \int\left(\frac{\overset{\cdot\cdot\cdot}{\underset{n-1}{B}}}{\overset{\cdot}{c}^3}\right)\overset{\prime\prime\prime\prime}{c} +, \&c. \dots (7),$$

or, if we consider $\underset{n-1}{B}$ under the form $\underset{n-1}{B} = \beta + \overset{(1)}{\beta}\overset{\prime}{c} + \overset{(2)}{\beta}\overset{\prime}{c^2} + \overset{(3)}{\beta}\overset{\prime}{c^3} +,$

$$\underset{n}{B} = \int\overset{\prime}{\underset{n-1}{B}}\overset{\prime}{c} + 1\int\overset{(1)}{\beta}\overset{\prime\prime}{c} + 1.2\int\overset{(2)}{\beta}\overset{\prime\prime\prime}{c} + 1.2.3\int\overset{3)}{\beta}\overset{\prime\prime\prime\prime}{c} +, \&c. \dots (8)*$$

Where any number of strokes under the β's denotes that the fluxional coefficients of $f(c)$ therein contained, must be depressed so many orders.

In this expression, we must neglect in $\overset{(2)}{\beta}$ all terms containing $\overset{\prime\prime}{c}$, in $\overset{(3)}{\beta}$ all those containing $\overset{\prime\prime}{c}$ or $\overset{\prime\prime\prime}{c}$, and so on. Let it be required, for an example, to find the coefficient of x^7, from that of x^6 given in article 3. We shall have, after neglecting such terms as are above specified,

$$\overset{(1)}{\beta} = \overset{\prime\prime}{f}(c)\overset{\prime\prime\prime\prime}{c} + \overset{\prime\prime}{f}(c)\overset{\prime\prime}{c}\overset{\prime\prime\prime}{c}; \quad \overset{(2)}{\beta} = \overset{\prime\prime\prime}{f}(c)\frac{\overset{\prime\prime\prime}{c}}{2}; \quad \overset{(6)}{\beta} = \overset{\prime\prime\prime\prime}{f}(c)\frac{1}{2.3\,4.5.6},$$

and by performing the operations indicated by equation (8), we find

$$\int\overset{\prime}{\underset{6}{B}}\overset{\prime}{c} = \overset{\prime\prime}{f}(c)\overset{\prime}{c}\overset{\prime\prime\prime\prime\prime}{c} + \overset{\prime\prime}{f}(c)\begin{vmatrix}\frac{\overset{\prime}{c^2}}{2}\overset{\prime\prime\prime\prime}{c}\\+\overset{\prime}{c}\overset{\prime\prime}{c}\overset{\prime\prime\prime\prime}{c}\\+\overset{\prime}{c}\frac{\overset{\prime\prime\prime}{c^2}}{2}\end{vmatrix} + \overset{\prime\prime\prime}{f}(c)\begin{vmatrix}\frac{\overset{\prime}{c^3}}{2.3}\overset{\prime\prime\prime}{c}\\+\frac{\overset{\prime}{c^2}}{2}\overset{\prime\prime}{c}\overset{\prime\prime\prime}{c}\\+\overset{\prime}{c}\frac{\overset{\prime\prime}{c^3}}{2.3}\end{vmatrix} + \overset{\prime\prime\prime\prime}{f}(c)\begin{vmatrix}\frac{\overset{\prime}{c^4}}{2.3.4}\overset{\prime\prime\prime}{c}\\+\frac{\overset{\prime}{c^3}}{2.3}\frac{\overset{\prime\prime}{c^2}}{2}\end{vmatrix} + \overset{\prime\prime\dots 6}{f}(c)\frac{\overset{\prime}{c^5}}{2.3.4.5}\overset{\prime\prime}{c} + \overset{\prime\prime\dots 7}{f}(c)\frac{\overset{\prime}{c^7}}{2.3.4\dots7}$$

* See Note III. at the end.

$$\int_{\bullet}^{\bullet(1)} \overset{''}{\beta}\,\overset{''}{c\cdot} = \overset{''}{f}(c)\,\overset{''}{c}\,\overset{''''}{c} + \overset{'''}{f}(c)\,\frac{\overset{''}{c^2}}{2}\,\overset{'''}{c}; \quad 2\int_{\bullet}^{\bullet(2)} \overset{'''}{\beta}\,\overset{'''}{c\cdot} = \overset{''}{f}(c)\,\overset{''}{c}\,\overset{''''}{c};$$

$$2.3.4.5.6\int_{\bullet}^{\bullet(6)} \underset{''''}{\beta}\,\overset{''\dots7}{c}\cdot = \overset{'}{f}(c)\,\overset{''\dots7}{c}; \text{ these being added give } \overset{B}{7} \text{ the}$$

coefficient of x^7, which was required.

To find $\overset{B}{n}$ from equation (7) requires the use of both fluxions and fluents; in (8) we are without the fluxional process; but, in its place, have the trouble of observing the numeral coefficients of each term: there is, however, a way of avoiding the mention of fluents, and the necessity of paying attention to these coefficients. If we consider equation (3) and the mode of expansion derived from it, it will be evident, that whenever we have any power (as the mth) of $\overset{'}{c}$ or $\overset{''}{c}$ or $\overset{'''}{c}$, or &c. it must be divided by the product $2.3.4\dots m$. This follows from the manner of finding the fluent of such a quantity as $\overset{'r}{c}\,\overset{'}{c\cdot}$, and the consideration that we cannot arrive at $\overset{,m}{c}$ without having passed through all the lower powers, and repeated the fluential process at each. Hence results the following rule; (where by $\overset{(2)}{\beta}$, $\overset{(3)}{\beta}$, &c. I mean these quantities after we have neglected in each of them the terms that have been already specified).

Omitting all the denominators, multiply $\underset{n-1}{\overset{\prime}{B}}$ *by* $\overset{'}{c}$; $\overset{(1)}{\beta}$ *by* $\overset{''}{c}$; $\underset{\prime}{\overset{(2)}{\beta}}$ *by* $\overset{'''}{c}$; $\underset{''}{\overset{(3)}{\beta}}$ *by* $\overset{''''}{c}$; *and so on: add these products together, and wherever there is any power of* $\overset{'}{c}, \overset{''}{c}$, *&c. as the* mth, *put the product* $2.3.4\dots m$ *for a denominator.*

6. *Third Method.* If we substitute in equation (3) the

values of $\overset{\prime}{\underset{n-2}{B}}$, $\overset{\prime}{\underset{n-3}{B}}$, &c. given by (4), and it will become

$$\underset{n}{B} = \int\left(\frac{\overset{\cdot}{\underset{n-1}{B}}}{\overset{\cdot}{c}}\right)\overset{\cdot}{c} + \int\left(\frac{\overset{\cdot}{\underset{n-1}{B}}}{\overset{\prime}{\overset{\cdot}{c}}}\right)\overset{\prime\cdot}{c} + \int\left(\frac{\overset{\cdot}{\underset{n-1}{B}}}{\overset{\prime\prime}{\overset{\cdot}{c}}}\right)\overset{\prime\prime\cdot}{c} + \int\left(\frac{\overset{\cdot}{\underset{n-1}{B}}}{\overset{\prime\prime\prime}{\overset{\cdot}{c}}}\right)$$

$\overset{\prime\prime\prime\prime\cdot}{c} +$, &c. ...... (9),

where we must neglect, in the second term, every thing that contains $\overset{\prime}{c}$; in the third term every thing that contains $\overset{\prime}{c}$ or $\overset{\prime\prime}{c}$; and so on.

If we did not do this, we should have the same combinations of letters, frequently, more than once. We may, however, instead of proceeding according to the above given direction, omit the superfluous terms *at last;* and then the rule will be as follows :

To find $\underset{n}{B}$, *take the fluxion of* $\underset{n-1}{B}$ *with respect to* $c, \overset{\prime}{c}, \overset{\prime\prime}{c}, $ &c. *and after changing, every where,* $\overset{\prime\prime\ldots m}{c} \cdot$ *into* $\overset{\prime\prime\ldots(m+1)}{c}\cdot$, *take the fluent with respect to this last; observing to keep only once the same combination of letters.*

But now let us consider, whether we cannot, by omitting to make certain of the letters vary, prevent the same combinations from being repeated.

First, if, in the term $P \times \left(\overset{\prime\prime\ldots(r-m)}{c}\right)^{n} \times \left(\overset{\prime\prime\ldots r}{c}\right)^{q}$, we make $\overset{\prime\prime\ldots(r-m)}{c}$ vary, according to the rule just now given, there results the combination $P \times\left(\overset{\prime\prime\ldots(r-m)}{c}\right)^{n-1} \times \overset{\prime\prime\ldots(r-m+1)}{c} \times \left(\overset{\prime\prime\ldots r}{c}\right)^{q} \ldots . (\alpha);$ but the same fluxional coefficient of $f(c)$ that is multiplied by $P \times \left(\overset{\prime\prime\ldots(r-m)}{c}\right)^{n} \times \left(\overset{\prime\prime\ldots r}{c}\right)^{q}$ will be also multiplied by $P \times\left(\overset{\prime\prime\ldots(r-m)}{c}\right)^{n-1}$

$\times \overset{''...(r-m+1)}{c} \quad \times \overset{''...(r-1)}{c} \quad \times \left(\overset{''...r}{c}\right)^{q-1}$, (where $\overset{''...(r-1)}{c}$ is either the last factor, with respect to the number of strokes, or the last but one, accordingly as q is equal to or greater than one); if we make this term vary, with respect to $\overset{''...(r-1)}{c}$, we shall have the combination marked (α) over again.

Let us next consider when it will be necessary to make c vary. In the term $\overset{''...n}{f}(c) \times P \times \left(\overset{''...m}{c}\right)^{p}$..... ($\beta$), if we make c vary, there arises the combination $\overset{''...(n+1)}{f(c)} \times \overset{'}{c} \times P \times \left(\overset{''...m}{c}\right)^{p}$; but the same coefficient (as B) that contains (β), will also contain $\overset{''...(n+1)}{f(c)} \times \overset{'}{c} \times \overset{r}{P} \times \overset{''...(m-1)}{c} \times \left(\overset{''...m}{c}\right)^{p-1}$ which when we make it vary with respect to $\overset{''...(m-1)}{c}$ gives also $\overset{''...(n+1)}{f(c)} \times \overset{'}{c} \times$ $P \times \left(\overset{''...m}{c}\right)^{p}$. Now $\overset{''...(m-1)}{c}$ was here the last quantity or the last but one. We may then affirm, in general, that, if we make every term in B vary with respect to the last quantity, and the last but one $\overset{n-1}{\text{also}}$, when this immediately precedes the last, not in place only, but in the number of its strokes, we shall get all the terms we ought to have, any further variation only giving the same over again. From equation (.9) we have then the following

Rule.

To find B *take the fluxion of* B *with respect to the last of the* quantities $\overset{n}{c}, \overset{'}{c}, \overset{''}{c}$, &c. *in each term, and the last but one $\overset{n-1}{\text{also}}$, if it immediately precede the last in the number of strokes.*

Change every where $c^{''...m}$ *into* $c^{''...(m+1)}$ *and take the fluent with respect to this last.*

This is exactly the same rule as that given by M. ARBOGAST in p. 25 of his work.

7. The method pursued in this paper, has a remarkable advantage over M. ARBOGAST's in what he calls *inverse derivation;* [*] which I shall shew hereafter to be extremely useful in the expansion of double and triple, &c, multinomials. In the present case, of a simple one, we have, as was shewn at the end of Art. 3,

$$B_{n-1} = \left(\frac{\dot{B}}{n}\right)_1; \quad B_{n-2} = \left(\frac{\dot{B}}{n}\right)_1; \quad B_{n-3} = \left(\frac{\dot{B}}{n}\right)_1; \quad \ldots\ldots B_{n-m} = \left(\frac{\dot{B}}{n}\right)_1;$$

whence this

Rule.

To find B_{n-m} *, the coefficient of* x^{n-m} *, from* B_n *that of* x^n *, take the fluxional coefficient of the latter, with respect to* $c^{''...m}$ *, and at the same time depress, to the next lower order, all the fluxional coefficents of* $f(c)$ *that are in* $\left(\dfrac{\dot{B}_n}{c^{''...,m}}\right)$ *.*

Thus from the coefficient of x^6, which was found in Art. 3, we get

$$B_5 = \left(\frac{\dot{B}}{6}\right)_1 = f'(c)\,\dot{c}''' + f''(c)\left|\dot{c}'\dot{c}''' \atop + \dot{c}''\dot{c}''' \right. + f'''(c)\left|\frac{\dot{c}'^2}{2}\dot{c}''' \atop + \dot{c}'\frac{\dot{c}''^2}{2}\right. + f''''(c)\frac{\dot{c}'^3}{2.3}\dot{c}'' + f'''''(c)\frac{\dot{c}'^5}{2.3.4.5}$$

$$\underset{4}{B} = \begin{pmatrix} \overset{\overset{\bullet}{B}}{6} \\ \overset{''}{c} \end{pmatrix}_{1} = f'(c)\,\overset{''''}{c} + f''(c)\begin{vmatrix} \overset{'}{c}\,\overset{'''}{c} \\ + \dfrac{\overset{''}{c^2}}{2} \end{vmatrix} + \overset{'''}{f}(c)\,\dfrac{\overset{'}{c^2}}{2}\,\overset{''}{c} + \overset{''''}{f}(c)\,\dfrac{\overset{'}{c^4}}{2\cdot3\cdot4}$$

&c.   &c.

8. To complete the theory of the expansion of any function of a simple multinomial, there remains, for us to solve, the following

Problem.

It is required to find $\underset{n}{B}$ *without knowing any of the coefficients that precede or follow it.*

It is, in the first place, evident enough, from what has been done, that

$$\underset{n}{B} = f'(c)\,\overset{''...n}{c} + f''(c)\,\overset{''}{\psi} + f'''(c)\,\overset{'''}{\psi} + \ \ldots\ldots\ + \ \overset{''...(m-1)}{f}(c)\,\overset{''...(m-1)}{\psi}$$

$$+ f(c)\,\overset{''...m}{\psi} + \ \ldots\ldots + f(c)\,\dfrac{\overset{'n}{c}}{2\cdot3\cdot4\ldots n},$$

where $\overset{''...m}{\psi}$ consists (without considering the denominators) of all the combinations that can be formed of $\overset{'}{c}, \overset{''}{c}, \overset{'''}{c}$, &c. in which the sum of the strokes shall be n,* and the sum of the exponents m. But to form these combinations, for the higher powers, would not be very easy. It may not be amiss to inquire, therefore, for some regular method of immediately deriving $\overset{''...(m-1)}{\psi}$ from $\overset{''...m}{\psi}$; so that we may get all the ψ's successively, beginning with $\dfrac{\overset{'n}{c}}{2.3...n}$ which multiplies $f(c)$.

---

* I mean when the powers are expanded, as when c^3 is written $\overset{'}{c}\,\overset{'}{c}\,\overset{'}{c}.$

I shall take no notice of any numbers, which divide the different terms, till the end of the operation; having shewn, in Article 5, that it will be sufficient then to place the product $2.3.4....\mu$ under every μth power.

There can be no difficulty in perceiving, that all the combinations in $\psi^{''...(m-1)}$ may be derived from those terms in $\psi^{''...m}$, that are multiplied by the powers of $\overset{,}{c}$, in the following manner. First, diminish the exponent of $\overset{,}{c}$ by one. Then, diminish the exponent of one of the other quantities by one, and multiply by the quantity that has the next greater number of strokes. For, if $\left(c^{''...r}\right)^a \times P \times \left(c^{''...s}\right)^b$ be one of the combinations in $\psi^{''...(m-1)}$, there must necessarily be in $\psi^{''...m}$ the combination $\overset{,}{c} \times \left(c^{''...r}\right)^a \times P \times c^{''...(s-1)} \times \left(c^{''...s}\right)^{b-1}$; and from this the former one is derived, in the manner above-mentioned, by taking away the $\overset{,}{c}$ and changing $c^{''...(s-1)}$ into $c^{''...s}$.

We will next see if there be any quantities that it would be superfluous to make vary. Let $c^{,b} \times P \times \left(c^{''...(p-q)}\right)^r \times \left(c^{''...p}\right)^s$ be a term of $\psi^{''...m}$, we find from it, by the prescribed operation, the two terms of $\psi^{''...(m-1)}$

$$c^{,b-1} \times P \times \left(c^{''..(p-q)}\right)^r \times \left(c^{''...p}\right)^{s-1} \times c^{''...(p+1)} \text{ and } c^{,b-1} \times P \times \left(c^{''..(p-q)}\right)^{r-1}$$

$$\times c^{''...(p-q+1)} \times \left(c^{''...p}\right)^s \dots (\alpha);$$ but $\psi^{''....m}$ will also have the combination $c^{,b} \times P \times \left(c^{''...(p-q)}\right)^{r-1} \times c^{''...(p-q+1)} \times c^{''...(p-1)} \times \left(c^{''...p}\right)^{s-1}$, from which the combination (α) may be got by diminishing by

one the exponent of $\overset{,}{c}$ and changing $\overset{''...(p-1)}{c}$ into $\overset{''...p}{c}$. Now $\overset{''...(p-1)}{c}$ *is either the last quantity or the last but one in the order of the strokes.*

We can then have no difficulty in perceiving the truth of the following

Rule.*

1st. *To find* $\overset{''...(m-1)}{\psi}$ *take no notice of any terms in* $\overset{''...m}{\psi}$ *but those that are multiplied by the powers of* $\overset{,}{c}$, *in all these diminish the exponent of* $\overset{,}{c}$ *by one; and omit the denominators.*

2dly. *Diminish the exponent of the last quantity, in these terms, by one; and multiply by the quantity that has the next greater number of strokes.*

3dly. *If the last quantity but one be that which immediately precedes the last in the number of strokes, make it vary in the same manner as was directed for the last.*

4thly. *All the combinations being thus formed, put the product* $2.3.4....\mu$ *under every μth power.*

The reader may compare this rule with that given by M. ARBOGAST, p. 36.

Suppose it were required to find B; we must begin with $\underset{10}{}$

$$\overset{''...10}{\psi} = \frac{\overset{,10}{c}}{2.3.4...10}; \text{ from which is derived by the rule, } \overset{''...9}{\psi} = \frac{\overset{,8}{c}}{2.3...8} \times \overset{''}{c},$$

$$\overset{''...8}{\psi} = \frac{\overset{,7}{c}}{2.3...7} \times \overset{'''}{c} + \frac{\overset{,6}{c}}{2.3...6} \times \frac{\overset{''2}{c}}{2}, \text{ and so on, whence}$$

$$\underset{10}{B} = \overset{''...10}{f(c)} \frac{\overset{,10}{c}}{2.3.4...10} + \overset{''...9}{f(c)} \frac{\overset{,8}{c}}{2.3.4...8} \overset{''}{c} + \overset{''...8}{f(c)} \left\{ \frac{\overset{,7}{c}}{2.3.4...7} \overset{'''}{c} + \frac{\overset{,6}{c}}{2.3.4...6} \frac{\overset{''2}{c}}{2} \right\}$$

$+, \&c.$

* See Note I. at the end.

The succeeding terms are found with equal ease, I omit to find them only on account of the length of the calculation.

9. I shall now show that the same method may be success-fully employed in more complicated cases; and, instead of dwelling on particular problems, shall proceed at once to *the expansion of any function of any functions of simple multino-mials,*

$$\varphi\left\{ F\left(c + \overset{\prime}{c}\,x + \overset{\prime\prime}{c}\,x^2 +\right), f\left(e + \overset{\prime}{e}\,x + \overset{\prime\prime}{e}\,x^2 +\right), \&c.\right\} \ldots\ldots(\alpha).$$

If we consider $c + \overset{\prime}{c}\,x +$, $e + \overset{\prime}{e}\,x +$, &c. as binomials $c + y$, $e + z$, &c. the function (α), which may be, for the moment, represented by φ, will have for its fluxion, (y, z, &c. being made to vary)

$$\left(\frac{\dot\varphi}{\dot y}\right)\dot y + \left(\frac{\dot\varphi}{\dot z}\right)\dot z + \&c. = \left(\frac{\dot\varphi}{\dot c}\right)\dot y + \left(\frac{\dot\varphi}{\dot e}\right)\dot z + \&c.\,;\text{ and conse-}$$

quently $\int\left\{\left(\frac{\dot\varphi}{\dot c}\right)\dot y + \left(\frac{\dot\varphi}{\dot e}\right)\dot z + \&c.\right\} = \varphi = \varphi\left\{F(c + \overset{\prime}{c}\,x + \overset{\prime\prime}{c}\,x^2 +),\right.$

$\left. f(e + \overset{\prime}{e}\,x + \overset{\prime\prime}{e}\,x^2 +), \&c.\right\}\ldots(\beta).$

If then we represent the expansion of the function (α) by the series

$$B + \underset{1}{B}\,x + \underset{2}{B}\,x^2 + \underset{3}{B}\,x^3 + \ldots\ldots + \underset{n}{B}\,x^n + \ldots\ldots(\gamma)$$

and denote the fluxional coefficients, of the first order of B, $\underset{1}{B}$, $\underset{2}{B}$, &c. with respect to c thus $\overset{c}{B}$, $\overset{c}{\underset{1}{B}}$, $\overset{c}{\underset{2}{B}}$, &c.;

with respect to e thus $\overset{e}{B}$, $\overset{e}{\underset{1}{B}}$, $\overset{e}{\underset{2}{B}}$, &c.;

&c. &c.

the equation marked (β) will become

$$\varphi\left\{F(c + \overset{\prime}{c}\,x +), f(e + \overset{\prime}{e}\,x +), \&c.\right\} = \int\left\{\overset{c}{B} + \overset{c}{\underset{1}{B}}\,x + \overset{c}{\underset{2}{B}}\,x^2 +\right\}$$

$$(\dot{c}\dot{x}+2\ddot{c}x\dot{x}+)+\int\left\{\overset{e}{B}+\overset{e}{\underset{1}{B}}x+\overset{e}{\underset{2}{B}}x^2+\right\}(\dot{e}\dot{x}+2\ddot{e}x\dot{x}+)+\&c.$$

whence, after taking the fluents, and comparing the coefficients of the powers of x, with those of the same powers in (γ), we find

$$\underset{1}{B}=\overset{c}{\dot{c}\,B}+\overset{e}{\dot{e}\,B}+\&c.$$

$$\underset{2}{B}=\frac{2\,\overset{c}{\ddot{c}\,B}+\overset{c}{\dot{c}\,\underset{1}{B}}}{2}+\frac{2\,\overset{e}{\ddot{e}\,B}+\overset{e}{\dot{e}\,\underset{1}{B}}}{2}+\&c.$$

$$\underset{3}{B}=\frac{3\,\overset{c}{\dddot{c}\,B}+2\,\overset{c}{\ddot{c}\,\underset{1}{B}}+\overset{c}{\dot{c}\,\underset{2}{B}}}{3}+\frac{3\,\overset{e}{\dddot{e}\,B}+2\,\overset{e}{\ddot{e}\,\underset{1}{B}}+\overset{e}{\dot{e}\,\underset{2}{B}}}{3}+\&c.$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$\underset{n}{B}=\frac{\overset{''\ldots n\,c}{n\,c\;B}+\overset{''\ldots(n-1)\,c}{(n-1)\,c}\underset{1}{B}+\ldots\ldots+2\,\overset{c}{\ddot{c}\,\underset{n-2}{B}}+\overset{c}{\dot{c}\,\underset{n-1}{B}}}{n}$$

$$+\frac{\overset{''\ldots n\,e}{n\,e\;B}+\overset{''\ldots(n-1)\,e}{(n-1)\,e}\underset{1}{B}+\ldots\ldots+2\,\overset{e}{\ddot{e}\,\underset{n-2}{B}}+\overset{e}{\dot{e}\,\underset{n-1}{B}}}{n}+\&c.\;\ldots\ldots\;(\delta).$$

But $B=\varphi\left\{F(c),f(e),\&c.\right\}$ whence all the coefficients are known.

10. This solution, however, gives no very expeditious way of actually expanding the function in question; particularly when we get to the higher powers: but by proceeding as in the second method, made use of for a function of one single multinomial, we find

$$\underset{n}{B}=\int\overset{c}{\underset{n-1}{B}}\dot{c}+\int\overset{c}{\underset{n-2}{B}}\ddot{c}+\int\overset{c}{\underset{n-3}{B}}\dddot{c}+\ldots\ldots+\int\overset{e}{\underset{n-1}{B}}\dot{e}+\int\overset{e}{\underset{n-2}{B}}\ddot{e}+\int\overset{e}{\underset{n-3}{B}}\dddot{e}+$$

$$\ldots\ldots+\&c.\;\ldots\ldots\;(\varepsilon)$$

where we must neglect in $\overset{c}{\underset{n-2}{B}}$ all terms which contain c; in $\overset{c}{\underset{n-3}{B}}$

all those which contain $\overset{\prime}{c}$ or $\overset{\prime\prime}{c}$; and so on. In $\overset{e}{\underset{n-1}{\mathrm{B}}}$ must be neg-

lected all terms which contain any of the c's; in $\overset{e}{\underset{n-2}{\mathrm{B}}}$ these, and

those also containing $\overset{\prime}{e}$; and, *in general, all those terms must be neglected, as we proceed, which contain any quantities whose flux-ions have entered into the preceding terms.*

From the above equation is derived an easy mode of expansion, I shall give an example in the case of two functions; and shall represent, for brevity, $\varphi\left\{\mathrm{F}\,(c),\, f\,(e)\right\}$ by φ, and its fluxional coefficient of the $\overline{m+n}$th order (when the fluxion has been taken m times with respect to c and n times with respect to e) by $\overset{m,n}{\varphi}$.

We find here, $\mathrm{B}=\varphi$; $\underset{1}{\mathrm{B}}=\overset{1,0}{\varphi}\,\overset{\prime}{c}+\overset{0,1}{\varphi}\,\overset{\prime}{e}$;

$\underset{2}{\mathrm{B}}=\overset{1,0}{\varphi}\,\overset{\prime\prime}{c}+\overset{2,0}{\varphi}\,\frac{\overset{\prime2}{c}}{2}+\overset{0,1}{\varphi}\,\overset{\prime\prime}{e}+\overset{0,2}{\varphi}\,\frac{\overset{\prime2}{e}}{2}+\overset{1,1}{\varphi}\,\overset{\prime}{c}\,\overset{\prime}{e}$; but to explain more fully the manner of proceeding, let it be required to find $\underset{3}{\mathrm{B}}$ from the preceding coefficients. We have, after neglecting the specified terms,

$$\int^{c}\underset{2}{\mathrm{B}}\,\overset{\prime}{c}=\overset{2,0}{\varphi}\,\overset{\prime}{c}\,\overset{\prime\prime}{c}+\overset{3,0}{\varphi}\,\frac{\overset{\prime}{c^3}}{2.3}+\overset{1,1}{\varphi}\,\overset{\prime}{c}\,\overset{\prime\prime}{e}+\overset{1,2}{\varphi}\,\overset{\prime}{c}\,\frac{\overset{\prime}{e^2}}{2}+\overset{2,1}{\varphi}\,\frac{\overset{\prime}{c^2}}{2}\,\overset{\prime}{e};\quad \int^{c}\underset{1}{\mathrm{B}}\,\overset{\prime\prime}{c}=\overset{1,1}{\varphi}\,\overset{\prime\prime}{c}\,\overset{\prime}{e}$$

$$\int^{c}\mathrm{B}\,\overset{\prime\prime\prime}{c}=\overset{1,0}{\varphi}\,\overset{\prime\prime\prime}{c};\quad \int^{e}\underset{2}{\mathrm{B}}\,\overset{\prime}{e}=\overset{0,2}{\varphi}\,\overset{\prime}{e}\,\overset{\prime\prime}{e}+\overset{0,3}{\varphi}\,\frac{\overset{\prime}{e^3}}{2.3};\quad \int^{e}\mathrm{B}\,\overset{\prime\prime\prime}{e}=\overset{0,1}{\varphi}\,\overset{\prime\prime\prime}{e};$$ the sum of these gives

$$\underset{3}{\mathrm{B}}=\overset{1,0}{\varphi}\,\overset{\prime\prime\prime}{c}+\overset{2,0}{\varphi}\,\overset{\prime}{c}\,\overset{\prime\prime}{c}+\overset{3,0}{\varphi}\,\frac{\overset{\prime}{c^3}}{2.3}+\overset{0,1}{\varphi}\,\overset{\prime\prime\prime}{e}+\overset{0,2}{\varphi}\,\overset{\prime}{e}\,\overset{\prime\prime}{e}+\overset{0,3}{\varphi}\,\frac{\overset{\prime}{e^3}}{2.3}+\overset{1,1}{\varphi}\,(\overset{\prime}{c}\,\overset{\prime\prime}{e}+\overset{\prime\prime}{c}\,\overset{\prime}{e})$$

$$+\overset{1,2}{\varphi}\,\overset{\prime}{c}\,\frac{\overset{\prime}{e^2}}{2}+\overset{2,1}{\varphi}\,\frac{\overset{\prime}{c^2}}{2}\,\overset{\prime}{e}.$$

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11. To get methods of deriving any coefficient from the one immediately preceding it, we must substitute, in (δ) and (ϵ), the values of $\overset{c}{\underset{n-m}{B}}$ and $\overset{e}{\underset{n-m}{B}}$ given by the following equations, which are similar to those found before, in the case of one multinomial, and marked (4) and (5).

$$\underset{n-m}{\overset{c}{B}} = \left(\frac{\overset{\cdot}{B}}{\underset{c}{\overset{n}{''...m}}}\right) = \left(\frac{\overset{\cdot}{B}}{\underset{c}{\overset{n-1}{''...(m-1)}}}\right); \quad \underset{n-m}{\overset{e}{B}} = \left(\frac{\overset{\cdot}{B}}{\underset{e}{\overset{n}{''...m}}}\right) = \left(\frac{\overset{\cdot}{B}}{\underset{e}{\overset{n-1}{''...(m-1)}}}\right);$$

&c. .... (η)

$$\underset{n-m}{\overset{c}{B}} = \left(\frac{\overset{m-1}{\overset{\cdot}{B}}}{\underset{c}{'^{m-1}}}\right)_{cc....(m-2)}; \quad \underset{n-m}{\overset{e}{B}} = \left(\frac{\overset{m-1}{\overset{\cdot}{B}}}{\underset{e}{'^{m-1}}}\right)_{ee....(m-2)}; \&c. ...(\theta)$$

where, by any number of c's or e's placed under a quantity, I represent the depression of the fluxional coefficients of φ, contained in that quantity, so many orders, with respect to c or e.

By combining equations (η) and (δ), we find

$$n\underset{n}{B} = n\overset{''...n}{c}\left(\frac{\overset{\cdot}{B}}{\underset{e}{''.(n-1)}}\right) + (n-1)\overset{''...(n-1)}{c}\left(\frac{\overset{\cdot}{B}}{\underset{c}{''...(n-2)}}\right) + ...$$

$$+2\overset{''}{c}\left(\frac{\overset{\cdot}{B}}{\underset{c}{'}}\right) + \overset{'}{c}\left(\frac{\overset{\cdot}{B}}{\overset{\cdot}{c}}\right) + n\overset{''...n}{e}\left(\frac{\overset{\cdot}{B}}{\underset{e}{''...(n-1)}}\right) + (n-1)\overset{''...(n-1)}{e}$$

$$\left(\frac{\overset{\cdot}{B}}{\underset{e}{''...(n-2)}}\right) + ... + 2\overset{''}{e}\left(\frac{\overset{\cdot}{B}}{\underset{e}{'}}\right) + \overset{'}{e}\left(\frac{\overset{\cdot}{B}}{\overset{\cdot}{e}}\right) + \&c. \text{ which in}$$

words is this: *To find* $\underset{n}{B}$, *take the fluxion of* $\underset{n-1}{B}$ *with respect to*

$c, c', c'', \&c.\ e, e', e'', \&c.\ \&c.$ *and change, every where,* $c'^{\cdots m}$ *into*

$c''^{\cdots(m+1)} \times \frac{m+1}{n}$, *and* $e''^{\cdots m}$ *into* $e''^{\cdots(m+1)} \times \frac{m+1}{n}, \&c.$

The reader may try this rule on the examples in the last article: I proceed to simpler methods for practice.

By combining equations (θ) and (ε), and considering $\underset{n-1}{B}$ under the forms $\underset{n-1}{B} = \beta + \overset{(1)}{\beta}' c + \overset{(2)}{\beta}' c^{\bullet} + \overset{(3)}{\beta}' c^{\bullet} +$, and $\underset{n-1}{B} = \Delta +$

$\overset{(1)}{\Delta}' e + \overset{(2)}{\Delta}' e^{\bullet} + \overset{(3)}{\Delta}' e^{\bullet} +$, and $\&c.$ we find $\underset{n}{B} = \int_{n-1}^{c} \overset{}{B}' c^{\bullet} + \int^{\overset{(1)}{}} \overset{(1)}{\beta}'' c^{\bullet} +$

$2\int_{c}^{\overset{(2)}{}} \overset{(2)}{\beta}''' c^{\bullet} + 2.3\int_{cc}^{\overset{(3)}{}} \overset{(3)}{\beta}''' c^{\bullet} + ... + \int_{n-1}^{e} \overset{}{B}' e^{\bullet} + \int^{\overset{(1)}{}} \overset{(1)}{\Delta}'' e^{\bullet} + 2\int_{e}^{\overset{(2)}{}} \overset{(2)}{\Delta}''' e^{\bullet} + 2.3$

$\int_{ee}^{\overset{(3)}{}} \overset{(3)}{\Delta}''' e^{\bullet} + ... + \&c.$ where in $\overset{(2)}{\beta}$ all terms must be neglected

which contain c''; and in general, according the rule given in article 10, all terms must be neglected that contain any quantities whose fluxions have entered into the preceding terms.

By this method the expansion might be accomplished without difficulty, each term is found at once, and no reductions are necessary: the one which I am going to give is, however, much better, being, I conceive, the simplest possible.

12. By combining the equations (η) and (ε), there results

$$\underset{n}{B} = \int\int\left(\frac{\dot{B}}{\underset{\dot{c}}{n-1}}\right) c^{\bullet} + \int\int\left(\frac{\dot{B}}{\underset{\dot{c}'}{n-1}}\right)' c^{\bullet} + \int\int\left(\frac{\dot{B}}{\underset{\dot{c}''}{n-1}}\right)'' c^{\bullet} + ...$$

$$+ \int\int\left(\frac{\dot{B}}{\underset{\dot{e}}{n-1}}\right)' e^{\bullet} + \int\int\left(\frac{\dot{B}}{\underset{\dot{e}'}{n-1}}\right)'' e^{\bullet} + \int\int\left(\frac{\dot{B}}{\underset{\dot{e}''}{n-1}}\right)''' e^{\bullet} + ... + , \&c.$$

where we must observe to neglect certain terms, according to the directions so often given: and if we apply here all that was said in article 6, when we were considering a similar expression for a function of one multinomial, we easily get the following

Rule.

To find B_n *from* B_{n-1} *, in the expansion of a function of any functions of the multinomials,* $c + c'x + c''x^2 +$ *and* $e + e'x + e''x^2 + and\ d + d'x + d''x^2 + and\ \&c.$

1st. Consider only the c's, and take the fluxion of B_{n-1} *, with respect to the last of them in each term; and the last but one also, if it immediately precede the last in the number of its strokes: change, every where,* $c^{''...m}$ *into* $c^{''...(m+1)}$ *, and take the fluent of each term with respect to this last.*

2dly. Neglect all terms in B_{n-1} *which contain* $c', c'', c''', \&c.$ *and proceed, with the remaining ones, in the same manner with respect to the e's.*

3dly. Neglect all terms in B_{n-1} *which contain* $c', c'', c''', \&c.\ e', e'', e''',$ *&c. and proceed, with the remaining ones, in the same manner with respect to the d's.—And so on.*

Let it be required, in the case of two multinomials, to find B_4 from B_3 which is given in article 10. The first part of the rule gives

$$\varphi^{1,0}_{\;\;\;'''}c + \varphi^{2,0}\left(c'c''' + \frac{c''^2}{2}\right) + \varphi^{3,0}\frac{c'^2}{2}c'' + \varphi^{4,0}\frac{c'^4}{2.3.4} + \varphi^{1,1}c'e'' + \varphi^{2,1}\frac{c'^2}{2}e'' + \varphi^{1,1}c'''e'$$

$$+ \overset{1,2}{\varphi} c \overset{''}{\frac{e^2}{2}} + \overset{2,2}{\varphi} \frac{c^2}{2} \overset{'}{\frac{e^2}{2}} + \overset{2,1}{\varphi} c \overset{''}{c} \overset{'}{e} + \overset{3,1}{\varphi} \frac{c^3}{2.3} \overset{'}{e} + \overset{1,1}{\varphi} c \overset{'''}{e} + \overset{1,2}{\varphi} c \overset{''}{e} \overset{''}{e} + \overset{1,3}{\varphi} c \overset{'}{\frac{e^3}{2.3}}.$$

The second part gives

$$\overset{0,1}{\varphi} \overset{''''}{e} + \overset{0,2}{\varphi} \left(\overset{'}{e} \overset{'''}{e} + \overset{''}{\frac{e^2}{2}} \right) + \overset{0,3}{\varphi} \overset{'}{\frac{e^2}{2}} \overset{''}{e} + \overset{0,4}{\varphi} \overset{'}{\frac{e^4}{2.3.4}}.$$

The sum of these is $\underset{4}{B}$. As the number of multinomials adds nothing to the difficulty of expansion, according to this method, it is useless to give more examples.

13. Nor does the number of multinomials make any difference as to the facility of *inverse derivation;* which depends on the equation

$$\underset{n-m}{B} = \left(\frac{\overset{\cdot}{\underset{n}{B}}}{\overset{''\ldots m}{c} \cdot} \right)_c$$

Thus from $\underset{4}{B}$, just now given, in the case of two multinomials, let it be required to find $\underset{2}{B}$; we have

$$\underset{2}{B} = \underset{4-2}{B} = \left(\frac{\overset{\cdot}{\underset{4}{B}}}{\overset{''}{c} \cdot} \right)_c = \overset{1,0}{\varphi} \overset{''}{c} + \overset{2,0}{\varphi} \frac{c^2}{2} + \overset{0,1}{\varphi} \overset{''}{e} + \overset{0,2}{\varphi} \overset{'}{\frac{e^2}{2}} + \overset{1,1}{\varphi} \overset{'}{c} \overset{'}{e}.$$

14. There remains the important

Problem.

To find $\underset{r}{B}$ without knowing any of the other coefficients.

It will be plain to any one, who in the least considers the methods that have been employed, that $\underset{r}{B}$ must contain all the possible combinations of $\overset{'}{c}, \overset{''}{c}, \overset{'''}{c}$, &c. $\overset{'}{e}, \overset{''}{e}, \overset{'''}{e}$, &c. $\overset{'}{d}, \overset{''}{d}, \overset{'''}{d}$, &c. &c. that can be formed with this condition, that the number of strokes be r. Every mth power will be divided by the product

2.3.4....m : and the fluxional coefficient * $\overset{\alpha,\,\beta,\,\gamma,\,\&c.}{\varphi}$, that multiplies any term, will have for the left hand figure over it that number which is the sum of the exponents of the c's; for the next figure on the same side that number which is the sum of the exponents of the e's; for the third that which is the sum of the exponents of the d's ; and so on.

The only difficulty then is to find these combinations (without the possibility of missing any, or the trouble of finding the same more than once) by some regular process of derivation.

A rule was given in Art. 8, when we were considering the similar problem in the case of one multinomial, for deriving all the combinations in B, in which the sum of the strokes is r, from $\overset{\prime r}{c}$ as origin of derivation.

The same rule will apply here, but instead of the one origin $\overset{\prime r}{c}$, we have, in the case of two multinomials, the origins

$$\overset{\prime r}{c},\ \overset{\prime r-1}{c}\,\overset{\prime}{e},\ \overset{\prime r-2}{c}\,\overset{\prime 2}{e},\ \dots\dots\dots\ \overset{\prime 2}{c}\,\overset{\prime r-2}{e},\ \overset{\prime}{c}\,\overset{\prime r-1}{e},\ \overset{\prime r}{e}.$$

Let us consider any particular origin as $\overset{\prime n}{c}\,\overset{\prime m}{e}$. I denote the term derived immediately from $\overset{\prime n}{c}$ (by the rule in Art. 8,) by $\Delta\,\overset{\prime n}{c}$; and the terms derived from this last, from the same rule by $\Delta^2\,\overset{\prime n}{c}$; those got from $\Delta^2\,\overset{\prime n}{c}$ by $\Delta^3\,\overset{\prime n}{c}$; and so on.

It is evident that all the possible combinations (of the kind

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*　$\overset{\alpha,\,\beta,\,\gamma,\,\&c.}{\varphi}$ represents the fluxional coefficient of $\varphi\left\{ F(c), f(e), V(d), \&c. \right\}$ of the order $\alpha+\beta+\gamma+$, &c. where the fluxion has been taken α times with respect to c, β times with respect to e, and γ times with respect to d; and so on.

we are seeking) derived from the origin $c'^n e'^m$, will be expressed by the product

$$(c'^n + \triangle c'^n + \triangle^2 c'^n + \triangle^3 c'^n +)(e'^m + \triangle e'^m + \triangle^2 e'^m + \triangle^3 e'^m +),$$

where each derivation of e'^m is multiplied by every one of c'^n, and conversely each one of c'^n by every one of e'^m.

We have nothing to do then but to deduce the derivations from c'^n and e'^m by the rule in Art. 8.

Suppose that B was the coefficient required, and that we wanted all the combinations arising from $c'^3 e'^2$. We have here

$$\triangle c'^3 = c' c''; \ \triangle^2 c'^3 = c'''; \ \triangle^3 c'^3 = 0; \ \triangle e'^2 = e''; \ \triangle^2 e'^2 = 0,$$

and, by substituting these values in the above product, we find all the combinations arising from the origin $c'^3 e'^2$ to be $c'^3 e'^2 +$

$$c'^3 e'' + c' c'' e'^2 + c''' e'^2 + c' c'' e'' + c''' e''.$$

It would be as well to write down the appropriate denominators to each combination as we proceed: and when we had treated all the origins of derivation in this manner, there would only remain to arrange the terms under their proper fluxional coefficients.*

---

* Instead of $c'^n e'^m$, I might have taken for origin of derivation $\varphi^{n,0} c'^n \times \varphi^{0,m} e'^m$; and after multiplying the factors

$$(\varphi^{n,0} c'^n + \varphi^{n-1,0} . \triangle c'^n + \varphi^{n-2,0} . \triangle^2 c'^n +)(\varphi^{0,m} e'^m + \varphi^{0,m-1} . \triangle e'^m + \varphi^{0,m-2} . \triangle^2 e'^m +)$$

have changed $\varphi^{v,0} \times \varphi^{0,\mu}$ into $\varphi^{v,\mu}$; but this would only give additional trouble without answering any useful end: it is sufficiently plain that the appropriate fluxional coefficient of $\triangle^v c'^n \times \triangle^\mu e'^m$ will be $\varphi^{n-v,m-\mu}$.

If the function under consideration contain three multinomials, the origins of derivation will be

$$\acute{c}^r \quad, \acute{c}^{r-1} \acute{e}, \acute{c}^{r-2} \acute{e}^2, \ldots\ldots\ldots \acute{c}^2 \acute{e}^{r-2}, \acute{c} \acute{e}^{r-1}, \acute{e}^r$$

$$\acute{d} \times (\acute{c}^{r-1}, \acute{c}^{r-2} \acute{e}, \acute{c}^{r-3} \acute{e}^2, \ldots\ldots\ldots \acute{c} \acute{e}^{r-2}, \acute{e}^{r-1})$$

$$\acute{d}^2 \times (\acute{c}^{r-2}, \acute{c}^{r-3} \acute{e}, \acute{c}^{r-4} \acute{e}^2, \ldots\ldots\ldots \acute{e}^{r-2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\acute{d}^{r-2} \times (\acute{c}^2 \quad, \acute{c} \acute{e} \quad, \acute{e}^2)$$

$$\acute{d}^{r-1} \times (\acute{c} \quad, \acute{e})$$

$$\acute{d}^r$$

and all the possible combinations derived from each particular origin as, $\acute{c}^n \acute{e}^m \acute{d}^p$, will be expressed by the product

$$(\acute{c}^n + \triangle \acute{c}^n + \triangle^2 \acute{c}^n +, \&c.) \quad (\acute{e}^m + \triangle \acute{e}^m + \triangle^2 \acute{e}^m +, \&c.)$$

$$(\acute{d}^p + \triangle \acute{d}^p + \triangle^2 \acute{d}^p +, \&c.)$$

The reader will easily extend the method, if necessary, to a greater number of multinomials.

As we have, in this manner, a certain, easy, and regular way of finding all the combinations in any coefficient B, the pro-blem is completely solved.

I go on to *multinomials of higher kinds:* and, with M. ARBOGAST, shall call those multinomials of the nth order which are disposed according to the powers and products of n different letters x, y, z, &c.

15. After having so fully entered into particulars, in the preceding cases, there can be no difficulty in perceiving, that *a complete theory of derivation, for the expansion of any function of any number of functions of multinomials, whether they be simple,*

or double, or triple, &c. is contained in the following equa-
tions.

$$\underset{m,n,r,\&c.}{B} = \Sigma.\int_{\cdot} \underset{m-\mu,n-\nu,r-\rho,\&c.}{\overset{c}{B}} \times \mu,\nu,\rho,\&c. + \Sigma.\int_{\cdot} \underset{m-\mu,n-\nu,r-\rho,\&c.}{\overset{\dot{c}}{B}}$$

$$\times \overset{\dot{e}}{\mu},\nu,\rho,\&c. + ,\&c. \dots\dots (\varkappa)$$

$$\underset{m-\mu,n-\nu,r-\rho,\&c.}{B} = \left(\frac{\overset{\dot{B}}{m,n,r,\&c.}}{\underset{\mu,\nu,\rho,\&c.}{\dot{c}}}\right)_{c} = \&c. \dots\dots (\lambda),$$

where $\underset{m,n,r,\&c.}{B}$ is the coefficient of $x^m y^n z^r$, &c. in the expan-

sion; $\underset{\mu,\nu,\rho,}{\overset{c}{}}$ &c. and $\underset{\mu,\nu,\rho,}{\overset{e}{}}$ &c. &c. are the coefficients of $x^\mu y^\nu z^\rho$
&c. before expansion, under the signs of the functions.

The sign Σ, in the first formula, expresses the sum of all
the terms that can be formed by taking for μ, ν, ρ, &c. all the
whole numbers from o to m, n, r, &c. μ, ν, ρ, &c. must not how-
ever all equal nothing at the same time.

It is scarcely necessary to observe, that certain terms are
understood to be neglected in equation ($\varkappa$), according to the
rule given in article 10, which is, that all terms in the B's must
be neglected, as we proceed, which contain quantities whose
fluxions enter into the preceding terms.

The above expressions if considered not only in themselves,
but with respect to the formulas that are immediately dedu-
cible from their developement and combination, in the manner
that will presently be shewn, appear to be the most general
and important in this branch of analysis.

16. Let it be required to expand the double multinomial
function

$$\varphi \begin{cases} c + c_{1,0}\, x + c_{2,0}\, x^2 + c_{3,0}\, x^3 + \\[4pt] + c_{0,1}\, y + c_{1,1}\, xy + c_{2,1}\, x^2 y + \\[4pt] + c_{0,2}\, y^2 + c_{1,2}\, xy^2 + \\[4pt] + c_{0,3}\, y^3 + \end{cases}$$

into the form

$$\begin{aligned} & B + B_{1,0}\, x + B_{2,0}\, x^2 + B_{3,0}\, x^3 + \\ & \quad + B_{0,1}\, y + B_{1,1}\, xy + B_{2,1}\, x^2 y + \\ & \qquad + B_{0,2}\, y^2 + B_{1,2}\, xy^2 + \\ & \qquad\qquad + B_{0,3}\, y^3 + \end{aligned}$$

Equation (κ) is in this case reduced to $B_{m,n} = \Sigma . \int_0^c B'_{m-\mu,n-\nu} \times \overset{c}{\mu,\nu}$ which expanded is

$$(\mu)\ \dots\dots\ B_{m,n} = \int_0^c B'_{m-1,n}\, \overset{c}{1,0} + \int_0^c B'_{m-2,n}\, \overset{c}{2,0} + \dots\dots\dots$$

$$+ \int_0^c B'_{m,n-1}\, \overset{c}{0,1} + \int_0^c B'_{m-1,n-1}\, \overset{c}{1,1} + \dots\dots\dots$$

$$+ \int_0^c B'_{m,n-2}\, \overset{c}{0,2} + \dots\dots\dots$$

$$+$$

By this formula, we find the first terms of the expansion to be

$$\varphi(c) + \varphi'(c)\, \overset{c}{1,0}\, x + \varphi'(c) \quad \overset{c}{2,0} \Big|\, x^2 +$$

$$+ \varphi'(c)\, \overset{c}{0,1}\, y + \varphi''(c) \quad \frac{\left(\overset{c}{1,0}\right)^2}{2} \Bigg|$$

$$+ \varphi'(c) \qquad \overset{c}{1,1} \Big|\, xy +$$

$$+ \varphi''(c)\, \overset{c}{1,0} \times \overset{c}{0,1} \Big|$$

$$+$$

Let it be required to find $B_{2,1}$. We have, after neglecting the terms specified,

$$\int \overset{\mathfrak{o}\,\prime}{\underset{1,1}{B}}\; \overset{\mathfrak{c}}{1,0} = \overset{\prime\prime}{\phi}(c)\, \overset{c}{1,0} \times \overset{c}{1,1} + \overset{\prime\prime\prime}{\phi}(c)\, \frac{\left(\overset{c}{1,0}\right)^2}{2} \times \overset{c}{0,1}\;;\quad \int \overset{\mathfrak{o}\,\prime}{\underset{2,0}{B}}\; \overset{\mathfrak{c}}{0,1} = \overset{\prime\prime}{\phi}(c)\, \overset{c}{0,1} \times \overset{c}{2,0}\;;$$

$$\int \overset{\mathfrak{o}\,\prime}{\underset{2,1}{B}}\; \overset{\mathfrak{c}}{2,1} = \overset{\prime}{\phi}(c)\, \overset{c}{2,1}\;;\;\text{ and these added together give}$$

$$\underset{2,1}{B} = \overset{\prime}{\phi}(c)\, \overset{c}{2,1} + \overset{\prime\prime}{\phi}(c)\left\{ \overset{c}{1,0} \times \overset{c}{1,1} + \overset{c}{0,1} \times \overset{c}{2,0} \right\} + \overset{\prime\prime\prime}{\phi}(c)\, \frac{\left(\overset{c}{1,0}\right)^2}{2} \times \overset{c}{0,1}.$$

I wrote the terms separately, and then collected them, for the better explanation of the method; but this double labour is by no means necessary: the coefficients may be formed and written down at once, as quickly as can be wished.

17. A very little consideration will convince us, that the terms $\displaystyle\int \underset{m,n-1}{\overset{\circ}{B}}\, \overset{\mathfrak{c}}{0,1}\,;\; \int \underset{m,n-2}{\overset{\circ}{B}}\, \overset{\mathfrak{c}}{0,2}\,;\; \cdots\cdots\; \int \underset{m,n-r}{\overset{\circ}{B}}\, \overset{\mathfrak{c}}{0,r}$ may be entirely left out of formula (μ), excepting when the term we search is of the form $\underset{0,m}{B}$, in which case it is reduced to the coefficient of y^m in the expansion of $\phi\left(c + \overset{c}{0,1}\,y + \overset{c}{0,2}\,y^2 +\right)$.

If then we neglect these terms, and, in the remaining ones, put for $\underset{m-\mu,n-\nu}{\overset{\prime}{B}}$ its value $\left(\dfrac{\overset{\circ}{\underset{m-1,n}{B}}}{\underset{\mu-1,\nu}{\overset{\mathfrak{c}}{}}}\right)$ given by (λ), equation (μ) will become

$$(\nu)\cdots\cdots\; \underset{m,n}{B} = \int \left(\dfrac{\overset{\circ}{\underset{m-1,n}{B}}}{\overset{\mathfrak{c}}{}}\right) \overset{\mathfrak{c}}{1,0} + \int \left(\dfrac{\overset{\circ}{\underset{m-1,n}{B}}}{\underset{1,0}{\overset{\mathfrak{c}}{}}}\right) \overset{\mathfrak{c}}{2,0} + \cdots\cdots\cdots$$

$$+ \int \left(\dfrac{\overset{\circ}{\underset{m-1,n}{B}}}{\underset{0,1}{\overset{\mathfrak{c}}{}}}\right) \overset{\mathfrak{c}}{1,1} + \cdots\cdots\cdots$$

$$\cdots\cdots\cdots + \int \left(\dfrac{\overset{\circ}{\underset{m-1,n}{B}}}{\underset{\mu-1,\nu}{\overset{\mathfrak{c}}{}}}\right) \overset{\mathfrak{c}}{\mu,\nu} + \cdots\cdots$$

by means of which equation we may find $B_{m,n}$ from $B_{m-1,n}$. Here, as usual, we must omit, in the successive fluxional coefficients, all terms containing quantities whose fluxions have been factors in the preceding terms of the formula. Let us, for an example, find $B_{3,1}$ from $B_{2,1}$ given in the last article; we have

$$\int \left(\frac{\dot{B}_{2,1}}{\dot{c}}\right) \dot{c}_{1,0} = \phi(c)\, c_{1,0} \times c_{2,1} + \phi''(c)\left\{\frac{(c_{1,0})^2}{2} \times c_{1,1} + c_{1,0} \times c_{0,1}\right.$$

$$\left. \times c_{2,0}\right\} + \phi'''(c)\frac{(c_{1,0})^3}{2.3} \times c_{0,1}; \quad \int \left(\frac{\dot{B}_{2,1}}{\dot{c}_{1,0}}\right)\dot{c}_{2,0} = \phi''_{1,1} \times c_{2,0}; \quad \int \left(\frac{\dot{B}_{2,1}}{\dot{c}_{2,0}}\right)$$

$$\dot{c}_{3,0} = \phi''(c)\, c_{0,1} \times c_{3,0}; \quad \int \left(\frac{\dot{B}_{2,1}}{\dot{c}_{2,1}}\right)\dot{c}_{3,1} = \phi'(c)\, c_{3,1}; \text{ whence } B_{3,1} =$$

$$\phi'(c)\, c_{3,1} + \phi''(c)\left\{c_{1,0} \times c_{2,1} + c_{1,1} \times c_{2,0} + c_{0,1} \times c_{3,0}\right\} + \phi'''(c)$$

$$\left\{\frac{(c_{1,0})^2}{2} \times c_{1,1} + c_{1,0} \times c_{0,1} \times c_{2,0}\right\} + \phi''''(c)\frac{(c_{1,0})^3}{2.3} \times c_{0,1}.$$

18. We may also derive from equation (ν) the following simple

Rule.

To find $B_{m,n}$, take the fluxion of $B_{m-1,n}$ with respect to all the quantities; change, every where, $\dot{c}_{\mu,\nu}$ into $\dot{c}_{\mu+1,\nu}$, and take the fluent with respect to this last. The same terms must be kept only once. c is $c_{0,0}$.

By this rule we frequently find the same terms more than once, which disadvantage is, however, more than compensated by its shortness, and the ease and simplicity of the process.

Let it be required from B to find successively $B_{0,3}$, $B_{1,3}$, $B_{2,3}$, &c.

We saw, when treating of a simple multinomial, that

$$B_{0,3} = \phi'(c)\, c_{0,3} + \phi''(c)\, c_{0,1} \times c_{0,2} + \phi'''(c)\, \frac{(c_{0,1})^3}{2\cdot 3}$$ whence, by the rule,

$$B_{1,3} = \phi'(c)\, c_{1,3} + \phi''(c)\left\{ c_{1,0} \times c_{0,3} + c_{1,1} \times c_{0,2} + c_{0,1} \times c_{1,2} \right\} + \phi'''(c)$$

$$\left\{ c_{1,0} \times c_{0,1} \times c_{0,2} + \frac{(c_{0,1})^2}{2} \times c_{1,1} \right\} + \phi''''(c)\, c_{1,0} \times \frac{(c_{0,1})^3}{2\cdot 3}\,;\quad B_{2,3} = \phi'(c)\, c_{2,3}$$

$$+ \phi''(c)\left\{ c_{2,0} \times c_{0,3} + c_{1,0} \times c_{1,3} + c_{2,1} \times c_{0,2} + c_{1,1} \times c_{1,2} + c_{0,1} \times c_{2,2} \right\}$$

$$+ \phi'''(c)\left\{ \frac{(c_{1,0})^2}{2} \times c_{0,3} + c_{1,0} \times c_{1,1} \times c_{0,2} + c_{1,0} \times c_{0,1} \times c_{1,2} + c_{2,0} \times c_{0,1} \right.$$

$$\left. \times c_{0,2} + c_{0,1}\frac{(c_{1,1})^2}{2} + \frac{(c_{0,1})^2}{2} \times c_{2,1} \right\} + \phi''''(c)\left\{ \frac{(c_{1,0})^2}{2} \times c_{0,1} \times c_{0,2} + c_{1,0} \right.$$

$$\left. \times \frac{(c_{0,1})^2}{2} \times c_{1,1} + c_{2,0} \times \frac{(c_{0,1})^3}{2\cdot 3} \right\} + \phi'''''(c)\, \frac{(c_{1,0})^2}{2} \times \frac{(c_{0,1})^3}{2\cdot 3}.$$ It is not ne-

cessary to calculate all the coefficients we may want by *direct derivation*; when we have got a few, in this manner, we may find the rest by the *inverse method* which is much easier.   M. ARBOGAST has put the twenty-eight first terms in a table;* of these there was need to calculate *only four directly*, as I shall show hereafter.   But, to give an example of this inverse proceeding, let it be required to find $B_{2,2}$ from $B_{2,3}$ just now given.

Equation (λ) becomes in the present case $B_{m-\mu,\,n-\nu} = \left(\dfrac{\dot{B}_{m,n}}{\dot{c}_{\mu,\nu}} \right)_{\prime}$;

whence $B_{2,2} = B_{2-0,\,3-1} = \left(\dfrac{\dot{B}_{2,3}}{\dot{c}_{0,1}} \right)_{\prime} = \phi'(c)\, c_{2,2} + \phi''(c)\left\{ c_{1,0} \times c_{1,2} \right.$

* *Calc. des Deriv.* p. 127.

$$+ \overset{c}{_{2,0}} \times \overset{c}{_{0,2}} + \frac{\left(\overset{c}{_{1,1}}\right)^2}{2} + \overset{c}{_{0,1}} \times \overset{c}{_{2,1}} \Big\} + \overset{'''}{\phi}(c)\Big\{\frac{\left(\overset{c}{_{1,0}}\right)^2}{2} \times \overset{c}{_{0,2}} + \overset{c}{_{1,0}} \times \overset{c}{_{0,1}}$$

$$\times \overset{c}{_{1,1}} + \overset{c}{_{2,0}} \times \frac{\left(\overset{c}{_{0,1}}\right)^2}{2} \Big\} + \overset{''''}{\phi}(c)\frac{\left(\overset{c}{_{1,0}}\right)^2}{2} \times \frac{\left(\overset{c}{_{0,1}}\right)^2}{2}.$$

19. Instead of leaving out of equation (μ) the terms of this form $\int \overset{'}{\underset{m,\,n-r}{\mathrm{B}}} \overset{c}{_{0,r}}$ as we did in article 17, we might have omitted those of the form $\int \overset{'}{\underset{m-r,\,n}{\mathrm{B}}} \overset{c}{_{r,0}}$; in which case it would have become

$$\underset{m,n}{\mathrm{B}} = \int \overset{'}{\underset{m,\,n-1}{\mathrm{B}}} \overset{c}{_{0,1}} + \int \overset{'}{\underset{m-1,\,n-1}{\mathrm{B}}} \overset{c}{_{1,1}} + \dots\dots$$

$$\int \overset{'}{\underset{m,\,n-2}{\mathrm{B}}} \overset{c}{_{0,2}} + \dots\dots$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$\dots\dots + \int \overset{'}{\underset{m-\mu,\,n-\nu}{\mathrm{B}}} \overset{c}{_{\mu,\nu}} + \dots$$

Here, if we put for $\overset{'}{\underset{m-\mu,\,n-\nu}{\mathrm{B}}}$ its value $\left(\dfrac{\overset{\cdot}{\underset{m+1,\,n-1}{\mathrm{B}}}}{\underset{\mu+1,\,\nu-1}{\overset{\cdot}{c}}}\right)$ derived from equation (λ), there results

$$\underset{m,n}{\mathrm{B}} = \int\!\!\int \left(\dfrac{\overset{\cdot}{\underset{m+1,\,n-1}{\mathrm{B}}}}{\underset{1,0}{\overset{\cdot}{c}}}\right) \overset{c}{_{0,1}} + \int\!\!\int \left(\dfrac{\overset{\cdot}{\underset{m+1,\,n-1}{\mathrm{B}}}}{\underset{2,0}{\overset{\cdot}{c}}}\right) \overset{c}{_{1,1}} + \dots\dots\dots$$

$$+ \int\!\!\int \left(\dfrac{\overset{\cdot}{\underset{m+1,\,n-1}{\mathrm{B}}}}{\underset{1,1}{\overset{\cdot}{c}}}\right) \overset{c}{_{0,2}} + \dots\dots\dots$$

$$\dots\dots\dots + \int\!\!\int \left(\dfrac{\overset{\cdot}{\underset{m+1,\,n-1}{\mathrm{B}}}}{\underset{\mu+1,\,\nu-1}{\overset{\cdot}{c}}}\right) \overset{c}{_{\mu,\nu}}$$

where, in every successive fluxional coefficient, certain terms are to be omitted, according to the usual rule.

Perhaps the simplest way of using this equation, although we shall frequently get the same combinations more than once, is by the following

Rule.

To find $B_{m,n}$ take the fluxion of $B_{m+1,\,n-1}$ with respect to all the quantities, excepting c, $c_{0,1}$, $c_{0,2}$, &c.; change every where, $\dot{c}_{\mu,\nu}$ into $\dot{c}_{\mu-1,\,\nu+1}$ and take the fluent with respect to this last. The same terms must be kept only once.

$B_{3,1}$ was found in article 17, from which we have, by this rule ;

$$B_{2,2} = \phi'(c)\ c_{2,2} + \phi''(c)\left\{ c_{0,1} \times c_{2,1} + c_{1,0} \times c_{1,2} + c_{0,2} \times c_{2,0} + \frac{\left(c_{1,1}\right)^2}{2} \right\}$$

$$+ \phi'''(c)\left\{ \frac{\left(c_{1,0}\right)^2}{2} \times c_{0,2} + \frac{\left(c_{0,1}\right)^2}{2} \times c_{2,0} + c_{1,0} \times c_{0,1} \times c_{1,1} \right\} + \phi''''(c)\ \frac{\left(c_{1,0}\right)^2}{2}$$

$$\times \frac{\left(c_{0,1}\right)^2}{2}.$$

Suppose that, beginning with $B_{6,0}$, we had calculated in this manner $B_{5,1}$, $B_{4,2}$, $B_{3,3}$; from these may be found, with the greatest ease, *and without any more direct derivation*, the twenty-eight first terms. For from $B_{5,1}$, $B_{4,2}$ we find $B_{1,5}$, $B_{2,4}$ merely by changing, in the former, the numbers that are on the right hand of the commas (under the c's) to the left hand, and the reverse. All the other terms are found, by inverse derivation, from the

equation $\quad B_{m-\mu,\,n-\nu} = \left(\dfrac{\dot{B}_{m,n}}{\dot{c}_{\mu,\nu}} \right)_{1}.$

<div align="center">*Problem.*</div>

20. *To find* $\underset{m,n}{B}$ *immediately, without knowing any of the other*

coefficients.

The coefficient of $x^m y^n$ will easily appear, from what has been shewn, to have the following form;

$$\underset{m,n}{B} = \overset{\prime}{\phi}(c)\underset{m,n}{\overset{c}{}} + \overset{\prime\prime}{\phi}(c)\overset{\prime\prime}{\psi} + \ldots + \overset{\prime\prime\ldots(r-1)}{\phi}(c)\overset{\prime\prime\ldots(r-1)}{\psi} + \overset{\prime\prime\ldots r}{\phi}(c)\overset{\prime\prime\ldots r}{\psi} + \ldots$$

$$+ \overset{\prime\prime\ldots(m+n)}{\phi}(c) \times \frac{\left(\underset{1,0}{c}\right)^m}{2.3\ldots m} \times \frac{\left(\underset{0,1}{c}\right)^n}{2.3\ldots n} \quad \text{where } \overset{\prime\prime\ldots r}{\psi} \text{ contains all the combina-}$$

tions that can be formed of the c's (after c or $\underset{0,0}{c}$) in which the sum of the bottom figures, on the left of the commas, is m; the sum of those on the right n: and the number of factors r. Moreover every power as the mth will be divided by $2.3\ldots m$.

And the reader, who considers how the similar problem was solved, in the case of a simple multinomial, will have no difficulty in perceiving the reason of the following very simple

<div align="center">*Rule.*</div>

To find $\overset{\prime\prime\ldots(r-1)}{\psi}$ *from* $\overset{\prime\prime\ldots r}{\psi}$, *1st. take the fluxional coefficient, of the latter, with respect to* $\underset{1,0}{c}$; *and, of this fluxional coefficient, take the fluxion with respect to all the quantities; change generally* $\underset{\mu,\nu}{\overset{\centerdot}{c}}$ *into* $\underset{\mu+1,\nu}{\overset{\centerdot}{c}}$, *and take the fluent with respect to this last.*

2dly. Any terms in $\overset{\prime\prime\ldots r}{\psi}$ *of the form* $\frac{\left(\underset{1,0}{c}\right)^p}{2.3\ldots p} \times \frac{\left(\underset{0,1}{c}\right)^q}{2.3\ldots q} \times \frac{\left(\underset{0,s}{c}\right)^t}{2.3\ldots t} \times \frac{\left(\underset{0,u}{c}\right)^v}{2.3\ldots v} \times \&c.$

where, except in $\underset{1,0}{c}$, *all the figures are on the right of the commas, will require, besides, the following process. Take the fluxional coefficient with respect to* $\underset{0,1}{c}$ *and, of this fluxional coefficient, take the*

fluxion with respect to all the quantities but $c_{1,0}$; *change generally* $c_{0,\mu}$ *into* $c_{0,\mu+1}$ *and take the fluent with respect to this last. The same terms must be kept only once.*

By this rule, we find $\underset{m,n}{B}$ beginning with $\varphi^{''...(m+n)}(c) \times \dfrac{\left(c_{1,0}\right)^m}{2.3...m} \times \dfrac{\left(c_{0,1}\right)^n}{2.3..n}$

for origin of derivation: the reader may compare it with that given by M. ARBOGAST at p. 113 of his work.

21. If the function to be expanded contains functions of many double multinomials, all the formulas, and rules, that have been given for one, may be extended to this case, by means of equations $(\varkappa)$ and (λ); in the same manner as a like extension was made in treating of simple multinomials.

Thus, from the method of finding $\underset{m,n}{B}$ given in article 19, we get the following

Rule.

To find $\underset{m,n}{B}$ *from* $\underset{m+1,\,n-1}{B}$ *in the expansion of any function of any functions of the double multinomials*

$$c + c_{1,0}\, x +\; ;\; e + e_{1,0}\, x +\; ;\; d + d_{1,0}\, x +\; ;\; \&c.$$
$$+ c_{0,1}\, y +\qquad + e_{0,1}\, y +\qquad + d_{0,1}\, y +$$
$$+\qquad\qquad\quad +\qquad\qquad\quad +$$

1st. *Consider only the c's, and take the fluxion of* $\underset{m+1,\,n-1}{B}$ *with respect to all of them except* c, $c_{0,1}$, $c_{0,2}$, *&c.; and proceed exactly in the same manner as was directed for one double multinomial in article* 19.

2dly. *Neglect all the terms in* $\underset{m+1,\,n-1}{B}$ *which contain any of the c's but c; and proceed, with the remaining terms, in the same manner with respect to the e's.*

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3dly. Neglect all terms in $\underset{m+1,\,n-1}{B}$ *which contain any of the c's or e's except c and e; and proceed, with the remaining terms, in the same manner with respect to the d's. And so on according to the number of multinomials.*

The sum of the terms, thus obtained, will give $\underset{m,n}{B}$.

It is scarcely necessary to observe that, when we have got a few of the higher terms, by this rule, the preceding ones may be found from the equation

$$\underset{m-\mu,\,n-\nu}{B} = \left(\frac{\overset{\cdot}{\underset{m,n}{B}}}{\underset{\mu,\nu}{\overset{\cdot}{c}}} \right)_c$$

as in the case of one double multinomial.

To find any coefficient, without a knowledge of the rest, when the function contains more than one double multinomial, we must combine the rule in the last article, with what was shewn in article 14.*

22. Thus we have a complete and simple theory of the expansion of functions of double multinomials; and from equations ($\varkappa$) and (λ) a precisely similar theory may be derived for multinomials of higher kinds.

But it is wholly unnecessary to enter into further details; we are able, without any more trouble, to see what must be the solution of the following

General Problem.

It is required, in the expansion of any function of a multinomial of any kind, to find $\underset{m,\,n,\,r,\,s,\,t,\,\&c.}{B}$ *the coefficient of* $x^m\,y^n\,z^r\,u^s\,v^t\,\&c.$

from $\underset{m+1,\,n-1,\,r,\,s,\,t,\,\&c.}{B}$ *that of* $x^{m+1}\,y^{n-1}\,z^r\,u^s\,v^t\,\&c.$

* See Note II.

Rule.

Take the fluxion of the latter with respect to all the quantities except c, $\overset{c}{_{0,\mu}}$, $\overset{c}{_{0,\mu,\nu}}$, $\overset{c}{_{0,0,\rho}}$, *&c that have nothing on the left hand of the first comma; change generally* $\overset{\dot c}{_{\mu,\nu,\rho,}}$ &c. *into* $\overset{\dot c}{_{\mu-1,\nu+1,\rho,}}$ &c. *and take the fluent with respect to this last. The same terms must be kept only once.*

The extension of this to any number of multinomials is exactly the same as the similar extension, for double multinomials, in the last article.

Second General Problem.

23. *It is required, in the case of the last problem, to find* $\underset{m,n,r,s,t,\&c.}{B}$ *without a knowledge of any other coefficient.*

This will be accomplished if we can find $\overset{''...(r-1)}{\psi}$ *which multiplies* $\overset{''...(r-1)}{\varphi}(c)$ *from* $\overset{''...r}{\psi}$ *which multiplies* $\overset{''..r}{\varphi}(c)$.

Rule.

1st. *Take the fluxional coefficient of* $\overset{''...r}{\psi}$ *with respect to* $\overset{c}{_{1,0,0,0,0,\&c.}}$, *and of this fluxional coefficient take the fluxion with respect to all the quantities; change generally* $\overset{\dot c}{_{\mu,\nu,\rho,}}$ &c. *into* $\overset{\dot c}{_{\mu+1,\nu,\rho,}}$ &c. *and take the fluent with respect to this last.*

2dly. *If there be any terms in* $\overset{''...r}{\psi}$ *in which the unit under* $\overset{c}{_{1,0,0,0,0,}}$ &c. *if it be one of the factors, is the only left hand figure, they will require a further process.*

Take the fluxional coefficient with respect to $\overset{c}{_{0,1,0,0,0,}}$ &c. *and of this take the fluxion with respect to all the quantities, except*

$\overset{c}{1,0,0,0,0}$, &c.; *change generally* $\overset{\dot c}{0,\mu,\nu,\rho}$, &c. *into* $\overset{\dot c}{0,\mu+1,\nu,\rho}$, &c. *and take the fluent.*

3*dly. Any terms, in* $\overset{''...r}{\psi}$, *in which the units under* $\overset{c}{1,0,0,0,0}$, &c. *and* $\overset{c}{0,1,0,0,0}$, &c., *if they are amongst the factors, are the only figures in the first and second left hand places, will require a still further process. Take the fluxional coefficient with respect to* $\overset{c}{0,0,1,0,0}$, &c., *and of this take the fluxion with respect to all the quantities except* $\overset{c}{1,0,0,0,0}$, &c. *and* $\overset{c}{0,1,0,0,0}$, &c.; *change generally* $\overset{\dot c}{0,0,\mu,\nu,\rho}$, &c. *into* $\overset{\dot c}{0,0,\mu+1,\nu,\rho}$, &c. *and take the fluent.*

The rule will proceed in this manner, till it contains n *parts if the multinomial be of the* n*th order. The terms arising from all these parts must be added, and the same terms kept only once.*

24. In treating multinomials of higher kinds, I have given rules by which certain terms are frequently found more than once: this was done for the sake of simplicity, and that the precepts might be easily retained in the memory; but was by no means a matter of necessity; for rules might without difficulty have been formed (as from equations (ν) and (ξ) for a double multinomial) by which no superfluous terms would have been found.

25. It will not be an improper termination of this paper, to state what are the peculiar advantages of the method pursued in it.

To many, I have no doubt, its brevity will be a recommendation; and that it requires no notation different from that in common use.

For though I have represented some of the fluxional coeffi-

cients in an unusual manner, as $\left(\dfrac{\overset{\cdot\cdot}{B}}{\underset{c^2.\dot{c}}{n-1}}\right)$ by $\left(\dfrac{\overset{\cdot\cdot}{B}}{\underset{c'^2}{n-1}}\right)^{c}$ the doing

so was not necessary; but it appeared advisable to make a distinction between the taking the fluxion with respect to c, and the same operation with respect to c', c'', &c. which enter into the coefficients in a manner different from the first.

The uniformity of the procedure is such, that, when we have arrived at the rules for one simple multinomial, a person of any skill in this kind of inquiry might easily divine those for the more difficult cases. But the most important circumstance is the perfection given to *inverse derivation*, and the facility with which we may, by that means, find any large number of terms in the expansion of the higher kinds of multinomials, as has been shewn in article 18 and 19.

The last advantage I shall notice is, that the same rules of derivation serve equally for the expansion of a function of one or of a thousand multinomials: whereas, from M. ARBOGAST's methods, it would not, I imagine, be very easy to give a rule in words for the expansion of a function of five or six.

NOTES.

Note I. The rule in article 8 may be differently enunciated thus.

To find $\overset{''\ldots(m-1)}{\psi}$ *take the fluxional coefficient of* $\overset{''\ldots m}{\psi}$ *with respect to* $\overset{\prime}{c}$; *and of this fluxional coefficient take the fluxion with respect to the last quantities; change generally* $\overset{''\ldots m}{c}$ *into* $\overset{''\ldots(m+1)}{c}$ *and take the fluent with respect to this last.*

2dly. If the last quantity but one be that which precedes the last in the number of strokes make it vary in the same manner and take the fluent.

This is simpler than the rule in article 8, and more conformable to the mode of expression made use of in other parts of the paper.

Note II. In looking back on what I have written, I am apprehensive it may be thought that I have affected too great brevity in the last paragraph of article 21. That the reader may have no difficulty, the following problem is added, to illustrate what was said in the passage alluded to.

Problem.

To find at once $\underset{m,n}{B}$ *in the expansion of a function of two functions of double multinomials.*

It is plain that $\underset{m,n}{B}$ must contain all the possible combinations of c's and e's (see the notation of article 21) that can be formed with this condition; that the number of left hand strokes be m; the number of right hand strokes n. Every rth power must be divided by the product $2.3.4\ldots r$. And the fluxional coefficient $\overset{\alpha,\beta}{\varphi}$, that multiplies each term, will have, for the left hand figure and over it, the sum of the exponents of the c's in that term; for the right hand figure β the sum of the exponents of the e's.

Now to get all the combinations of the kind mentioned above, with their proper divisors, we must plainly take, for origins of derivation, all the terms of the following product, when actually multiplied.

$$\left\{ \frac{\left(\overset{}{\underset{1,0}{c}}\right)^m}{2.3\ldots m} + \frac{\left(\overset{}{\underset{1,0}{c}}\right)^{m-1}}{2.3\ldots(m-1)} \times \underset{1,0}{e} + \frac{\left(\overset{}{\underset{1,0}{c}}\right)^{m-2}}{2.3\ldots(m-2)} \times \frac{\left(\overset{}{\underset{1,0}{e}}\right)^2}{2} + \cdots + \frac{\left(\overset{}{\underset{1,0}{e}}\right)^m}{2.3\ldots m} \right\}$$

$$\text{multiplied by} \left\{ \frac{\left(\overset{}{\underset{0,1}{c}}\right)^n}{2.3\ldots n} + \frac{\left(\overset{}{\underset{0,1}{c}}\right)^{n-1}}{2.3\ldots(n-1)} \times \underset{0,1}{e} + \frac{\left(\overset{}{\underset{0,1}{c}}\right)^{n-2}}{2.3\ldots(n-2)} \times \frac{\left(\overset{}{\underset{0,1}{e}}\right)^2}{2} + \cdots + \frac{\left(\overset{}{\underset{0,1}{e}}\right)^n}{2.3\ldots n} \right\}$$

Suppose any one of these origins to be

$$\frac{\left(\begin{smallmatrix}c\\1,0\end{smallmatrix}\right)^{r}}{2.3..r} \times \frac{\left(\begin{smallmatrix}c\\0,1\end{smallmatrix}\right)^{s}}{2.3...s} \times \frac{\left(\begin{smallmatrix}e\\1,0\end{smallmatrix}\right)^{t}}{2.3...t} \times \frac{\left(\begin{smallmatrix}e\\0,1\end{smallmatrix}\right)^{u}}{2.3...u} \dots\dots\dots (A).$$

Let Δ, Δ^2, Δ^3, &c. represent the successive derivations made according to the rule in article 20. It is plain that all the terms got from the origin of derivation (A) will be expressed by the product

$$\left\{ \frac{\left(\begin{smallmatrix}c\\1,0\end{smallmatrix}\right)^{r}}{2.3...r} \times \frac{\left(\begin{smallmatrix}c\\0,1\end{smallmatrix}\right)^{s}}{2\ 3...s} + \Delta \left\{ \frac{\left(\begin{smallmatrix}c\\1,0\end{smallmatrix}\right)^{r}}{2.3...r} \times \frac{\left(\begin{smallmatrix}c\\0,1\end{smallmatrix}\right)^{s}}{2.3...s} \right\} + \Delta^2 \left\{ \frac{\left(\begin{smallmatrix}c\\1,0\end{smallmatrix}\right)^{r}}{2.3...r} \times \frac{\left(\begin{smallmatrix}c\\0,1\end{smallmatrix}\right)^{s}}{2.3...s} \right\} + \&c. \right\}$$

multiplied by

$$\left\{ \frac{\left(\begin{smallmatrix}e\\1,0\end{smallmatrix}\right)^{t}}{2.3...t} \times \frac{\left(\begin{smallmatrix}e\\0,1\end{smallmatrix}\right)^{u}}{2.3...u} + \Delta \left\{ \frac{\left(\begin{smallmatrix}e\\1,0\end{smallmatrix}\right)^{t}}{2.3...t} \times \frac{\left(\begin{smallmatrix}e\\0,1\end{smallmatrix}\right)^{u}}{2.3...u} \right\} + \Delta^2 \left\{ \frac{\left(\begin{smallmatrix}e\\1,0\end{smallmatrix}\right)^{t}}{2.3...t} \times \frac{\left(\begin{smallmatrix}e\\0,1\end{smallmatrix}\right)^{u}}{2.3...u} \right\} + \&c. \right\}.$$

In this manner may the terms be derived from all the origins; after which we have only to arrange them under their appropriate fluxional coefficients.

If we wanted to find immediately $\underset{m,\ n,\ r,\ \&\text{c}.}{B}$ in a function of two multinomials of a still higher kind, the method would be exactly similar.

Note III. In the preceding pages, I have considered the expansion of multinomial functions *generally*; and abstained from giving particular examples, that the paper might not be extended to an unreasonable length. There are, however, some cases, —*when the function is a whole positive power*—which require a separate notice. The method of direct derivation given in article 5, and a similar one at the end of article 11 will here fail: this indeed is of no consequence, as the rules in article 6 and 12 are both easier than the former, and applicable to every case. But it will be necessary to give new methods of inverse derivation; for if we consider those in the paper, in article 7 for example, it will easily appear, that though they are true generally for the *m*th power, the case is very different when we give to this letter the particular values 1, 2, 3, &c. The reason of which is that the fluxional coefficients of $f(c)$, after the first, or the second, or the third, &c. vanish; and these functions may be said not improperly, when compared with the general form, to give *defective expansions*; any rules, therefore, which depend on the *depression* of the fluxional coefficients of $f(c)$ will be of no use here.

The following very extensive rule is the reverse of that, for direct derivation, in article 12. It agrees, in its *simplest case*, with that of M. ARBOGAST in his article 36.

Mr. KNIGHT *on the Expansion*

Rule.

To find B from B in a function of any functions of the multinomials $c + c' x +$
$n-1$ n

$c'' x^2 +, e + e' x + e'' x^2 +, d + d' x + d'' x^2 +,$ &c. 1st. Consider only those terms,
in B, which contain some of the quantities $c', c'', c''',$ &c.; reject all the terms in which
n

the last of these letters are raised to a higher than the first power: reject also *(if*
there be more than one multinomial) such terms as contain none of the above men-
tioned quantities but the first power of c'. Change, generally, in each remaining

term, the last of the c's as $\overset{'''\ldots m}{c}$ into $\overset{''\ldots(m-1)}{c}$ · and take the fluent with respect to this
quantity.

2dly. Neglecting those terms, in B, into which $c', c'', c''',$ &c. enter, consider those,
n

of the remainder, which contain $e', e'', e''',$ &c. rejecting all those terms in which the last
of the e's are raised to a higher than the first power. Those terms must also be rejected
(if there be more than two multinomials) which contain none of the e's but the first

power of e'. Change generally, in the remaining terms, the last of the e's as $\overset{''\ldots m}{e}$ into
$\overset{''\ldots(m-1)}{e}$ · and take the fluent, with respect to this last.

3dly. Neglecting the terms into which $c', c'', c''',$ &c. $e', e'', e''',$ &c. enter, consider those,
of the remainder, which contain $d', d'', d''',$ &c. and proceed as before.—And so on.

This rule has no difficulty, whatever may be the number of multinomials.

The words in italics, were inserted to make the rule include the finding of B from
B; they are of no use when n is greater than one.

Similar rules for multinomials of higher orders are formed with equal ease; being
the reverse of those that have been given for direct derivation.